Reduced state uniquely defines the Groverian measure of the original pure state

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Groverian and geometric entanglement measures of the n-party pure state are expressed by the $(n-1)$-party reduced state density operator directly. This main theorem derives several important consequences. First, if two pure n-qudit states have reduced states of $(n-1)$-qudits, which are equivalent under local unitary transformations, then they have equal Groverian and geometric entanglement measures. Second, both measures have an upper bound for pure states. However, this upper bound is reached only for two-qubit systems. Third, it converts effectively the nonlinear eigenvalue problem for the three-qubit Groverian measure into linear eigenvalue equations. Some typical solutions of these linear equations are written explicitly and the features of the general solution are discussed in detail.

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I. INTRODUCTION

Quantum theory opens up new possibilities for information processing and communication and the entanglement of a quantum state allows us to carry out tasks, which could not be possible with a classical system [1–8]. It plays a pivotal role for exponential speedup of quantum algorithms [9], teleportation [10], and superdense coding [11].

The quantum correlation is the essence of the entanglement and it cannot be created by local operations and classical communication (LOCC) alone. Analysis of multiparticle entanglement provides insight into the nature of quantum correlation. However, the current situation is far from satisfaction.

Linden *et al.* revealed that almost every pure state of three qubits is completely determined by its two-particle reduced density matrices [12]. In other words, we cannot get much new information from the given pure three-qubit state if the reduced two-qubit states are known. The case of pure states of any number n of parties was considered in Ref. [13] and it was shown that the reduced states of a fraction of the parties uniquely specify the quantum state. One may consider more general and open questions of vital importance: How much information is contained in any reduced $(n-1)$-qubit state? How do we use this information to convert the nonlinear eigenproblem of entanglement measure calculation to the linear eigenproblem? Is there any physically relevant connection between the pure n-party states which have local unitary (LU) equivalent $(n-1)$-party reduced states? Does such a connection impose an upper bound for entanglement measure?

Groverian entanglement measure G [14] gives concise answers to all of these questions. It is an entanglement measure defined in operational terms, namely, how well a given state serves as the input to Grover's search algorithm [15]. Groverian measure depends on maximal success probability $P_{\max}$ and is defined by the formula $G(\psi)=\sqrt{1-P_{\max}}$. The maximal success probability is the overlap of a given state with the nearest separable state. The same overlap defines geometric measure of entanglement introduced earlier as an axiomatic measure [16–18]. In this view Groverian measure gives an operational treatment of the axiomatic measure and is a good tool to investigate the above-mentioned questions. In the following we will consider only the maximal success probability and our conclusions are valid for both Groverian and geometric measures.

Surprisingly, any reduced state resulting from a partial trace over a single qubit suffices to find $P_{\max}$ of the original pure state. For example, the entanglement of the three-qubit pure state is completely understood from the two-qubit mixed state reduced from the original pure state. Since bipartite systems, regardless mixed or pure, always give a linear eigenproblem, this fact enables us to obtain analytic expressions of Groverian entanglement measures for pure three-qubit states.

It is well known that entanglement measures are invariant under local unitary transformations [4,19–21]. However, the LU-equivalent condition is not the only one for the same Groverian entanglement measure. In fact, if two pure states have LU-equivalent reduced states which are obtained by taking partial trace once, it turns out that they have the same entanglement measures. Owing to this the lower bound for $P_{\max}$ is derived. However, it is not reachable for three- and higher-qubit states and, therefore, is not precise.

In Sec. II we derive a formula connecting Groverian measure of a pure state and its reduced density matrix. In Sec. III we establish a lower bound for Groverian measure. In Sec. IV we present analytic expressions for the maximal success probability that reflect main features of both measures. In Sec. V we make concluding remarks.

II. GROVERIAN MEASURE IN TERMS OF REDUCED DENSITIES

We consider a pure n-qudit state $|\psi\rangle$. The maximum probability of success is defined by

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$$P_{\max}(\psi) = \max_{q_1 q_2 \cdots q_n} |\langle q_1 q_2 \cdots q_n|\psi\rangle|^2, \tag{1}$$

where $|q_k\rangle$'s are pure single-qudit normalized states. Our intention is to derive a formula which connects the maximum probability of success and $(n-1)$-qudit reduced states. In general, reduced states are mixed states and are described by density matrices. Hence, we express the maximum probability of success in terms of density operators right away. We will use the notation ρ for the state $|\psi\rangle$ and ϱ for the pure single-qudit state density operators, respectively. Equation (1) takes the form

$$P_{\max}(\rho) = \max_{\varrho_1 \varrho_2 \cdots \varrho_n} \mathrm{tr}(\rho \varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_n). \tag{2}$$

Theorem 1. Any $(n-1)$-qudit reduced state uniquely determines the Groverian and geometric measures of the original n-qudit pure state.

Proof. Define a single-qudit state $|\chi\rangle$ by the formula

$$|\chi\rangle = \langle q_1 q_2 \cdots \widehat{q_k} \cdots q_n|\psi\rangle, \tag{3}$$

where the caret means exclusion. Obviously

$$|\langle q_1 q_2 \cdots q_n|\psi\rangle|^2 = |\langle q_k|\chi\rangle|^2 = \mathrm{tr}(|\chi\rangle\langle\chi|\varrho_k). \tag{4}$$

The absolute value of the inner product $|\langle q_k|\chi\rangle|$ is maximum when $q_k = |\chi\rangle / \sqrt{\langle\chi|\chi\rangle}$ and therefore

$$\max_{\varrho_k} \mathrm{tr}(|\chi\rangle\langle\chi|\varrho_k) = \langle\chi|\chi\rangle = \mathrm{tr}(|\chi\rangle\langle\chi|). \tag{5}$$

Denote by $\rho(\hat{k})$ the reduced state resulting from a partial trace over the kth qudit, that is $\rho(\hat{k}) = \mathrm{tr}_k\, \rho(\psi)$. From this definition it follows the identity

$$\mathrm{tr}(|\chi\rangle\langle\chi|) = \mathrm{tr}[\rho(\hat{k})\varrho_1 \otimes \varrho_2 \otimes \cdots \widehat{\varrho_k} \cdots \otimes \varrho_n]. \tag{6}$$

Owing to this identity, Eq. (5) can be rewritten as

$$\max_{\varrho^k} \mathrm{tr}(\rho \varrho^1 \otimes \varrho^2 \otimes \cdots \otimes \varrho^n) = \mathrm{tr}[\rho(\hat{k})\varrho_1 \otimes \varrho_2 \otimes \cdots \widehat{\varrho_k} \cdots \otimes \varrho_n]. \tag{7}$$

Both sides of Eq. (7) must have the same maximum and this is the proof of the theorem.

Since the right-hand side of Eq. (7) contains the reduced density operator $\mathrm{tr}_k\, \rho = \rho(\hat{k})$ which is generally a mixed state, the next maximization is nontrivial.

Equation (7) does not mean that a pure state and its once reduced state have equal Groverian measures. One cannot maximize the mixed-state density matrix over product states to find the entanglement measure because the resulting measure is not an entanglement monotone [14,18,22].

Equation (7) connects directly the maximum probability of success with the reduced density operator

$$P_{\max}(\rho) = \max_{\varrho_1 \varrho_2 \cdots \widehat{\varrho_k} \cdots \varrho_n} \mathrm{tr}[\rho(\hat{k})\varrho_1 \otimes \varrho_2 \otimes \cdots \widehat{\varrho_k} \cdots \otimes \varrho_n]. \tag{8}$$

In fact, Theorem 1 is true for any entanglement measure. Consider an $(n-1)$-qudit reduced density matrix that can be purified by a single-qudit reference system. Let $|\psi'\rangle$ be any joint pure state. All other purifications can be obtained from the state $|\psi'\rangle$ by LU transformations $U \otimes \mathbb{1}^{\otimes(n-1)}$ where U is a local unitary matrix acting on single qudit and $\mathbb{1}$ is a unit matrix. Since any entanglement measure must be invariant under LU transformations, it must be the same for all purifications independently of U. Hence, the reduced density matrix ρ determines any entanglement measure on the initial pure state.

However, there is a crucial difference. In the case of Groverian measure the proof expresses entanglement measure by the reduced density matrix directly. As will be explained in Sec. IV, Eq. (8) is a simple and effective tool for calculating three-qubit entanglement measure. No such formula is known for other measures and general proof for other measures has limited practical significance.

Theorem 2. If two pure n-qudit states have LU equivalent $(n-1)$-qudit reduced states, then they have equal Groverian and geometric entanglement measures.

Proof. Assume that the density matrices of pure states are ρ and ρ' and corresponding maximum probabilities of success are $P_{\max}$ and $P'_{\max}$. Suppose the local unitary transformation $U^1 \otimes U^2 \otimes \cdots \otimes U^{n-1}$ maps $\rho'(\hat{k}') = \mathrm{tr}_{k'}\, \rho'$ to $\rho(\hat{k}) = \mathrm{tr}_k\, \rho$ as follows:

$$\rho(\hat{k}) = (U^1 \otimes U^2 \otimes \cdots \otimes U^{n-1})\rho'(\hat{k}') \times (U^1 \otimes U^2 \otimes \cdots \otimes U^{n-1})^+, \tag{9}$$

where the superscript plus sign means Hermitian conjugate. The trace with any complete product $\varrho^1 \otimes \varrho^2 \otimes \cdots \otimes \varrho^{n-1}$ state gives

$$\mathrm{tr}[\rho(\hat{k})\varrho^1 \otimes \varrho^2 \otimes \cdots \otimes \varrho^{n-1}] = \mathrm{tr}[\rho'(\hat{k}')\varrho'^1 \otimes \varrho'^2 \otimes \cdots \otimes \varrho'^{n-1}], \tag{10}$$

where $\varrho'^k = U^{k+}\varrho^k U^k$ are single-qubit pure states too. Let us choose the product state that maximizes the left-hand side. According to Eq. (8) the left-hand side is $P_{\max}$ and therefore $P_{\max} \le P'_{\max}$. Similarly $P'_{\max} \le P_{\max}$, therefore $P_{\max} = P'_{\max}$.

III. LOWER BOUND FOR MULTIQUBIT SYSTEMS

Theorem 1 sets a clear lower bound for the maximum probability of success.

Below A is an arbitrary 2×2 Hermitian matrix, $\boldsymbol{r}$ is a unit real three-dimensional vector, and components of the vector $\boldsymbol{\sigma}$ are Pauli matrices. The trace of the product of matrices A and $\boldsymbol{r} \cdot \boldsymbol{\sigma}$ can be presented as a scalar product of vectors $\boldsymbol{r}$ and $\mathrm{tr}(A\boldsymbol{\sigma})$. The scalar product of two real vectors with the constant modules is maximal when vectors are parallel. Consequently, we have

$$\max_{r^2=1} \mathrm{tr}(A\boldsymbol{r} \cdot \boldsymbol{\sigma}) = |\mathrm{tr}(A\boldsymbol{\sigma})| = \sqrt{(\mathrm{tr}\, A)^2 - 4 \det A} \tag{11}$$

and the positive root of radicals is understood.

An arbitrary density matrix ϱ for a pure state qubit may be written as $\varrho = 1/2(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma})$, where $\boldsymbol{r}$ is a unit real vector. Then Eq. (11) can be rewritten as

$$\max_{\varrho} \operatorname{tr}(A\varrho) = \frac{1}{2}[\operatorname{tr} A + \sqrt{(\operatorname{tr} A)^2 - 4 \det A}]. \tag{12}$$

From Eq. (12) it follows that

$$\max_{\varrho} \operatorname{tr}(A\varrho) \geq \frac{1}{2}(\operatorname{tr} A). \tag{13}$$

We define 2×2 matrix M_{n-1} by the formula

$$M_{n-1} = \operatorname{tr}_{1,2,\ldots,n-2}[\rho(\hat{n})\varrho^1 \otimes \varrho^2 \otimes \cdots \varrho^{n-2} \otimes \mathbb{1}], \tag{14}$$

where trace is taken over $(1,2,\ldots,n-2)$-qubits. Equation (8) takes the form

$$P_{\max} = \max_{\varrho^1\varrho^2\cdots\varrho^{n-1}} \operatorname{tr}(M_{n-1}\varrho^{n-1}), \tag{15}$$

where tr means trace over $(n-1)$-qubit. Equation (13) gives

$$\begin{aligned} P_{\max} &\geq \frac{1}{2} \max_{\varrho^1\varrho^2\cdots\varrho^{n-2}} \operatorname{tr} M_{n-1} \\ &= \frac{1}{2} \max_{\varrho^1\varrho^2\cdots\varrho^{n-2}} \operatorname{tr}[\rho(\hat{n})\varrho^1 \otimes \varrho^2 \otimes \cdots \varrho^{n-2} \otimes \mathbb{1}], \end{aligned} \tag{16}$$

where tr on right-hand side of Eq. (16) means trace over all qubits. Thus inequality (13) suggests a simple prescription: Replace a pure qubit density matrix by a unit matrix and add a multiplier 1/2 instead. We use this prescription $n-1$ times, eliminate all single-qubit density operators step by step from Eq. (8) and obtain

$$P_{\max} \geq \frac{1}{2^{n-1}}. \tag{17}$$

Note that this lower bound is valid only for pure states. The question at issue is whether it is a precise limit or not. If it is indeed the case, then what are the pure states which have the lower bound of $P_{\max}$? We will prove that this lower bound is reached only for bipartite states.

Denote by $\rho^{k_1k_2\cdots k_m}$ the reduced density operator of qubits $k_1k_2\cdots k_m$, $1\leq m\leq n-1$. Equation (7) and (13) together yield

$$P_{\max}(\rho) \geq \frac{1}{2^{n-m-1}} P_{\max}(\rho^{k_1k_2\cdots k_m}). \tag{18}$$

Note, $P_{\max}(\rho^{k_1k_2\cdots k_m})$ does not define any entanglement measure as $\rho^{k_1k_2\cdots k_m}$'s are mixed states. It is the maximal overlap of the mixed state with any product state and we use it as the intermediate mathematical quantity.

Lemma 2. If a pure state has limiting geometric and/or Groverian entanglement $P_{\max}=1/2^{n-1}$, then all of its reduced states are completely mixed states.

Proof. Equation (18) for $m=1$ and Eq. (12) impose

$$P_{\max} \geq \frac{1}{2^{n-1}}(1 + \sqrt{1 - 4 \det \rho^k}). \tag{19}$$

The maximal probability of success reaches the minimal value if the square root vanishes. Consequently, density matrices ρ^k must be a multiple of a unit matrix $\rho^k=\mathbb{1}/2$ and thus all one-qubit reduced states are completely mixed. Then two-qubit density matrices $\rho^{k_1k_2}$ must have the form

$$\rho^{k_1k_2} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + g_{\alpha\beta}\boldsymbol{\sigma}^\alpha \otimes \boldsymbol{\sigma}^\beta), \tag{20}$$

where $g_{\alpha\beta}=\operatorname{tr}(\rho^{k_1k_2}\boldsymbol{\sigma}^\alpha \otimes \boldsymbol{\sigma}^\beta)$ is a 3×3 matrix with real entries. Hereafter, the summation for repeated three-dimensional vector indices $(\alpha,\beta,\gamma,\ldots=1,2,3)$ is understood unless otherwise stated. To reach the lower bound we must have equality instead of inequality in (18) and this condition imposes $P_{\max}(\rho^{k_1k_2})=1/4$ resulting in $g_{\alpha\beta}=0$. Hence $\rho^{k_1k_2}=(1/4)\mathbb{1}\otimes\mathbb{1}$ and thus all two-qubit reduced states are completely mixed. One can continue this chain of derivations by induction. Indeed, suppose all m-qubit states $(m<n)$ are completely mixed. Then $(m+1)$-qubit density matrices $\rho^{k_1k_2\cdots k_{m+1}}$ must have the form

$$\begin{aligned} \rho^{k_1k_2\cdots k_{m+1}} = \frac{1}{2^{m+1}}(\mathbb{1}^{\otimes m+1} + g_{\alpha_1\alpha_2\cdots\alpha_{m+1}}\sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \\ \otimes \sigma^{\alpha_{m+1}}), \end{aligned} \tag{21}$$

where

$$g_{\alpha_1\alpha_2\cdots\alpha_{m+1}} = \operatorname{tr}(\rho^{k_1k_2\cdots k_{m+1}}\sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \otimes \sigma^{\alpha_{m+1}}). \tag{22}$$

From Eq. (18) it follows that $P_{\max}(\psi)$ takes its minimal value if $P_{\max}(\rho^{k_1k_2\cdots k_m})=1/2^m$. Equation (21) is consistent with this condition if and only if the maximization of the term of $g_{\alpha_1\alpha_2\cdots\alpha_{m+1}}\sigma^{\alpha_1}\otimes\sigma^{\alpha_2}\otimes\cdots\otimes\sigma^{\alpha_{m+1}}$ yields zero. Then $g_{\alpha_1\alpha_2\cdots\alpha_{m+1}}=0$ and therefore

$$\rho^{k_1k_2\cdots k_{m+1}} = \frac{1}{2^{m+1}}\mathbb{1}^{\otimes m+1}. \tag{23}$$

Thus if all m-qubit reduced states are completely mixed then all $(m+1)$-qubit reduced states are also completely mixed. On the other hand, all one-qubit reduced states are completely mixed. By induction all reduced states are completely mixed. The induction stops at pure states. In contrast to mixed states, the maximization of the term $g_{\alpha_1\alpha_2\cdots\alpha_n}\sigma^{\alpha_1}\otimes\sigma^{\alpha_2}\otimes\cdots\otimes\sigma^{\alpha_n}$ must yield unity for pure states as Eq. (7) requires.

Lemma is proved.

Theorem 3. None of the multiqubit pure states except two-qubit maximally entangled states satisfies the condition $P_{\max}=1/2^{n-1}$.

Proof. When $n=2$, it is well known that the EPR states and their LU-equivalent class reach the lower bound, i.e., $P_{\max}=1/2$. Now we would like to show that there is no pure state with limiting Groverian measure for $n=3$. Lemma 2 requires that the density matrix with limiting Groverian measure should be in the form

$$\rho = \frac{1}{8}(\mathbb{1}^{\otimes 3} + g_{\alpha\beta\gamma}\sigma^\alpha \otimes \sigma^\beta \otimes \sigma^\gamma). \tag{24}$$

Since ρ is a pure state density matrix, it must satisfy $\rho^2=\rho$. This condition leads to several constraints, one of which is

$$- i g_{\alpha\beta\gamma} g_{\delta\kappa\lambda} \epsilon_{\alpha\delta\delta'} \epsilon_{\beta\kappa\kappa'} \epsilon_{\gamma\lambda\lambda'} \sigma^{\delta'} \otimes \sigma^{\kappa'} \otimes \sigma^{\lambda'} = 6 g_{\alpha\beta\gamma} \sigma^{\alpha} \otimes \sigma^{\beta} \otimes \sigma^{\gamma} \quad (25)$$

where $\epsilon_{\alpha\beta\gamma}$ is an antisymmetric tensor. Since this constraint cannot be satisfied for real $g_{\alpha\beta\gamma}$, there is no pure state which has limiting Groverian measure at $n=3$.

Now we will show that there is no pure state for $n \geq 4$ too. Suppose there is n-qubit state $|\psi\rangle$ such that all of its reduced states are completely mixed. Choose a normalized basis of product vectors $|i_1 i_2 \cdots i_n\rangle$, where the labels within the ket refer to qubits $1,2,\ldots,n$ in that order. The vector $|\psi\rangle$ can be written as a linear combination

$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_n} C_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle \quad (26)$$

of vectors in the set. All reduced states of the state $|\psi\rangle$ are completely mixed if and only if

$$\sum_{i_k j_k} \delta_{i_k j_k} C_{i_1 i_2 \cdots i_n} C^*_{j_1 j_2 \cdots j_n} = \frac{1}{2^{n-1}} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \widehat{\delta_{i_k j_k}} \cdots \delta_{i_n j_n},$$

$$k = 1,2, \ldots ,n. \quad (27)$$

Note that the normalization condition follows from the above equation. Define $n-1$ index coefficients

$$D_{i_1 i_2 \cdots i_{n-1}} = \sqrt{2} C_{i_1 i_2 \cdots i_{n-1} 0}. \quad (28)$$

Setting $i_n = j_n = 0$ in Eq. (27) we obtain

$$\sum_{i_k j_k} \delta_{i_k j_k} D_{i_1 i_2 \cdots i_{n-1}} D^*_{j_1 j_2 \cdots j_{n-1}} = \frac{1}{2^{n-2}} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \widehat{\delta_{i_k j_k}} \cdots \delta_{i_{n-1} j_{n-1}},$$

$$k = 1,2, \ldots ,n-1. \quad (29)$$

Hence, the $(n-1)$-qubit state

$$|\phi\rangle = \sum_{i_1 i_2 \cdots i_{n-1}} D_{i_1 i_2 \cdots i_{n-1}} |i_1 i_2 \cdots i_{n-1}\rangle \quad (30)$$

exists and all of its reduced states are completely mixed. The contraposition of it is that if there is no pure state which has limiting Groverian measure at $n=3$, it is also true for $n \geq 4$. Theorem 3 is proved.

Thus, the lower bound of inequality (17) is unreachable for $n \geq 3$. This seems to mean that Eq. (17) is not a precise limit.

IV. ANALYTIC EXPRESSIONS FOR MAXIMUM PROBABILITY OF SUCCESS

The maximization of the pure three-qubit states over product states generally reduces to nonlinear eigenvalue equations [18]. However, Eq. (8) converts it effectively into linear eigenvalue equations. Thus, one can compute the entanglement measures for a wide range of three-qubit states analytically. As an illustration consider one parametric W-type [23] three-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{1+\kappa^2+\kappa^4}} (|100\rangle + \kappa|010\rangle + \kappa^2|001\rangle), \quad (31)$$

where κ is a free positive parameter. The calculation method is elaborated in Ref. [24] and here we present only final results. In three different ranges of definition the maximal success probability is differently expressed. In the first case, $P_{\max}$ is the square of the first coefficient provided it is greater than 1/2,

$$P_{\max} = \frac{1}{1+\kappa^2+\kappa^4}, \quad 0 < \kappa < \left(\frac{\sqrt{5}-1}{2}\right)^{1/2}. \quad (32)$$

In the second case, $P_{\max}$ is the square of the diameter of the circumcircle of the acute triangle formed by three coefficients,

$$P_{\max} = \frac{4\kappa^6}{(1+\kappa^2+\kappa^4)^2(3\kappa^2-1-\kappa^4)},$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^{1/2} \leq \kappa \leq \left(\frac{\sqrt{5}+1}{2}\right)^{1/2}. \quad (33)$$

In the third case, $P_{\max}$ is the square of the third coefficient provided it is greater than 1/2,

$$P_{\max} = \frac{\kappa^4}{1+\kappa^2+\kappa^4}, \quad \kappa > \left(\frac{\sqrt{5}+1}{2}\right)^{1/2}. \quad (34)$$

It is also possible to compute $P_{\max}$ for Eq. (31) numerically [25]. For numerical calculation we consider the kth qubit as $|q_k\rangle = \cos\theta_k|0\rangle + e^{i\varphi_k}\sin\theta_k|1\rangle$ with $k=1,2,3$. Since the coefficients of $|\psi\rangle$ are all real, we can set $\varphi_k=0$ for all k and express $P_{\max}$ in a form

$$P_{\max} = \max_{\theta_1,\theta_2,\theta_3} |\langle q_1|\langle q_2|\langle q_3|\psi\rangle|^2. \quad (35)$$

Thus numerical maximization over θ_1, θ_2, and θ_3 directly yields $P_{\max}$. As shown in Fig. 1(a) the numerical result (black dots) perfectly coincides with the analytic results (solid lines) expressed in Eqs. (32)–(34).

Let us consider another one parametric state

$$|\psi\rangle = \frac{1}{\sqrt{1+\kappa^2+\kappa^4+\kappa^6}} (|100\rangle + \kappa|010\rangle + \kappa^2|001\rangle + \kappa^3|111\rangle). \quad (36)$$

Again there are three cases. If four coefficients form a cyclic quadrilateral, then $P_{\max} = 4R^2$, where R is the circumradius of the quadrangle. Otherwise, $P_{\max}$ is the square of the largest coefficient. In the first case, $P_{\max}$ is the square of the first coefficient,

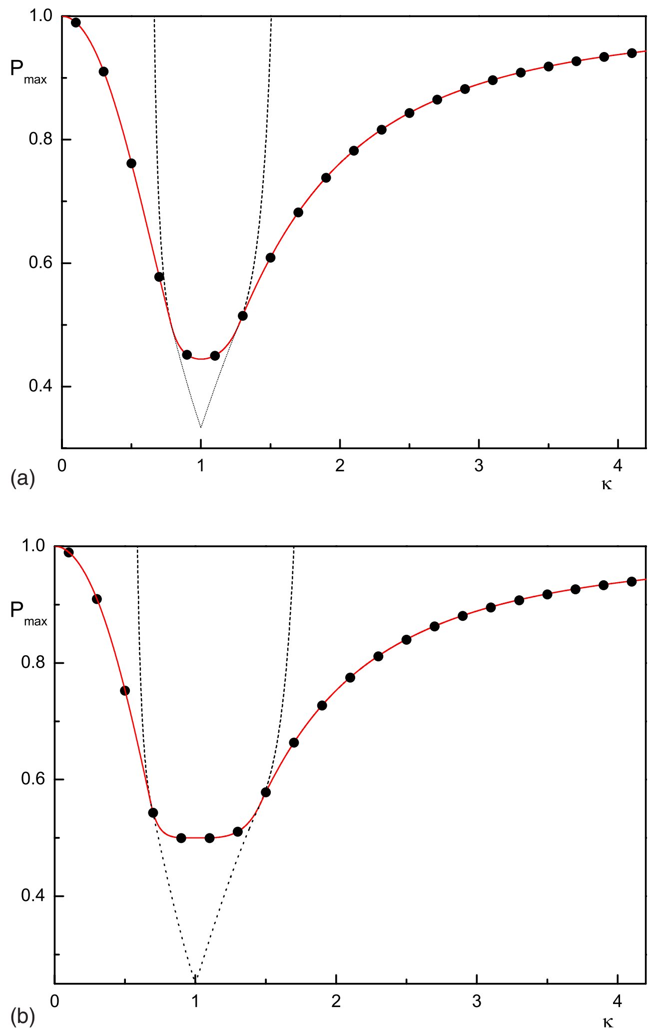

FIG. 1. (Color online) $P_{\max}$ for (a) Eq. (31) and (b) Eq. (36). The solid lines represent the analytical results of $P_{\max}$ and the black dots are the numerical results. This figure strongly supports that our analytical results are correct.

$$P_{\max}=\frac{1}{1+\kappa^2+\kappa^4+\kappa^6},$$

$$\kappa<\frac{1}{3}(\sqrt[3]{18\sqrt{57}+134}-\sqrt[3]{18\sqrt{57}-134}-1)^{1/2}\approx 0.685. \tag{37}$$

In the second case, $P_{\max}$ is the square of the circumcircle of the cyclic quadrangle formed by four coefficients,

$$P_{\max}=\frac{8\kappa^6}{-1+2\kappa^2+\kappa^4+8\kappa^6+\kappa^8+2\kappa^{10}-\kappa^{12}},$$

$$\frac{1}{3}(\sqrt[3]{18\sqrt{57}+134}-\sqrt[3]{18\sqrt{57}-134}-1)^{1/2}\leq\kappa$$
$$\leq\frac{1}{\sqrt{3}}(\sqrt[3]{46+6\sqrt{57}}+\sqrt[3]{46-6\sqrt{57}}+1)^{1/2}. \tag{38}$$

In the third case $P_{\max}$ is the square of the last coefficient,

$$P_{\max}=\frac{\kappa^6}{1+\kappa^2+\kappa^4+\kappa^6},$$

$$\kappa>\frac{1}{\sqrt{3}}(\sqrt[3]{46+6\sqrt{57}}+\sqrt[3]{46-6\sqrt{57}}+1)^{1/2}\approx 1.46. \tag{39}$$

The function $P_{\max}(k)$ and numerical results are shown in Fig. 1(b). Both figures strongly show that our analytical expressions of $P_{\max}$ perfectly coincide with the numerical result.

V. CONCLUSIONS

Equation (8) allows us to calculate the maximal success probability for three-qubit states which are expressed as linear combinations of four given orthogonal product states [26]. The answer is more complicated than a simple formula, but each final expression of the measure has its own meaningful interpretation. Namely, $P_{\max}$ can take the following values (up to numerical coefficients):

(i) The square of the circumradius of the cyclic polygon formed by coefficients of the state function.

(ii) The square of the circumradius of the crossed figure formed by coefficients of the state function.

(iii) The largest coefficient.

Each expression has its own range of definition where they are applicable. Although the above picture seems simple, the separation of the applicable domains is a highly nontrivial task. To make clear which of the expressions should be applied for a given state we refer to [26]. All of our results on Groverian measure of three-qubit pure states are summarized in [27].

Equation (8) gives the nonlinear eigenvalue problem for four- and higher-qubit states and it is natural to ask whether there is an extension of Eq. (8) that allows us to find analytic results for four-, five-, or general n-qubits. Although we have no distinct results here, we have obtained some insight from the analysis of the information contained in one- and two-qubit reduced states. Probably, it is possible to express the maximal success probability in terms of one- and two-qubit reduced states in case of four-qubit pure states. Such formula, if it can be derived, will give linear equations for four-qubit pure states. However, the situation is opposite in the case of five-qubit states. The method does now allow us to convert the task to the linear eigenvalue problem and more powerful tools are needed to calculate maximal success probability of general n-qubit states.

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[1] B. Schumacher, Phys. Rev. A **51**, 2738 (1995).
[2] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[4] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[5] M. Horodecki, Phys. Rev. A **57**, 3364 (1998).
[6] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[7] M. B. Plenio and V. Vedral, Contemp. Phys. **39**, 431 (1998).
[8] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[9] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[10] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[11] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[12] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. **89**, 207901 (2002).
[13] N. Linden and W. K. Wootters, Phys. Rev. Lett. **89**, 277906 (2002).
[14] O. Biham, M. A. Nielsen, and T. J. Osborne, Phys. Rev. A **65**, 062312 (2002).
[15] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
[16] A. Shimony, Ann. N.Y. Acad. Sci. **755**, 675 (1995).
[17] H. Barnum and N. Linden, J. Phys. A **34**, 6787 (2001).
[18] T.-C. Wei and P. M. Goldbart, Phys. Rev. A **68**, 042307 (2003).
[19] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
[20] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[21] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **84**, 2014 (2000).
[22] D. Shapira, Y. Shimoni, and O. Biham, Phys. Rev. A **73**, 044301 (2006).
[23] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[24] L. Tamaryan, D. K. Park, and S. Tamaryan, Phys. Rev. A **77**, 022325 (2008).
[25] Y. Shimoni, D. Shapira, and O. Biham, Phys. Rev. A **69**, 062303 (2004).
[26] L. Tamaryan, D. K. Park, J.-W. Son, and S. Tamaryan, e-print arXiv:0803.1040v1.
[27] E. Jung, M.-R. Hwang, D. K. Park, L. Tamaryan, and S. Tamaryan, e-print arXiv:0803.3311v1.